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Vertex Vulnerability in Wrapped Butterfly Network

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Abstract

Vulnerability is a critical factor in evaluating the topology of a network. A comprehensive understanding of these vulnerabilities is vital for enhancing the efficiency of network operations. In essence, an increase in a network's vulnerability correlates with a decrease in its security. The significance of the wrapped butterfly graph, derived from the family of cayley graphs, is evident in its strong symmetry and regularity, making it a highly efficient network with various attributes. This study not only focuses on a theoretical framework for detecting vulnerabilities, but also offers algorithms that tackle these vulnerabilities, centered on the wrapped butterfly graph as a model for interconnection networks. We conduct simulation experiments to assess the performance of these algorithms, exploring different types of vertex vulnerabilities and analyzing the overall robustness of interconnection networks. Identifying vulnerability parameters has been established as NP–complete across numerous graph categories. In this paper, we identify several vertex vulnerability parameters, including integrity, toughness, scattering number, tenacity, and rupture degree, related to the wrapped butterfly graph in polynomial time.

Keywords: combinatorial problems; vertex vulnerability parameters; wrapped butterfly graph; cut set; graph components.

1 Introduction and Background

Interconnection networks play a vital role in a wide array of applications in contemporary society. As reliance on interconnected systems grows across multiple domains, including power grids, transportation, and telecommunications, it becomes essential to scrutinize the vulnerabilities inherent in these intricate infrastructures. It has been observed that the interconnected characteristics of these systems can facilitate the rapid spread of failures, affecting extensive geographic areas [31].

Recent investigations have revealed that a particular scheme, proposed as part of established security standards, fell short of its intended purpose, failing to withstand sophisticated attacks. This vulnerability has exposed critical security flaws, underscoring the need for future research to address these concerns, particularly in the design of a new authenticated key agreement protocol [55]. Understanding a network's vulnerabilities is essential to assessing its security and resilience against potential threats. Examining weaknesses in RSA keys, in particular, helps uncover risks in encryption, authentication, and secure communication. Strengthening RSA key generation and management plays a key role in safeguarding networks from potential attacks [47]. In the context of graph theory to confront these challenges, researchers have introduced various vulnerability parameters to evaluate the resilience and robustness of interconnection networks. The vulnerability of these networks is primarily assessed through several critical parameters, including integrity, tenacity, toughness, scattering number, and rupture degree. The assessment of these parameters is crucial for understanding the network's ability to withstand failures, attacks and for the design of networks that ensure operational continuity even during disruptive events.

Graph theory provides a framework for examining the topological structure of interconnection networks by representing them as graphs [54]. In a communication network, nodes represent processors, while edges denote the communication channels. A significant disadvantage of a communication network is that it becomes extremely vulnerable to disruption if the failure of a few processors results in a large communication gap, and this study in the field of graph theory gave rise to the concept of integrity, i.e., the destruction caused by removing a few vertices from the graph, which in turn increases the component of the graph along with the study of the largest remaining components that are still active.

Integrity was suggested as a substitute metric for analyzing a graph's vulnerability to disruptions brought on by the removal of vertices [8]. The principle behind this was that integrity encompasses not just the removal of vertices, but also the consequences that follow, such as what happens to the largest connected component when some nodes are damaged or perform poorly. The concept of toughness introduces another dimension for evaluating graphs, offering a clear and concise measure of the interconnectivity among various graph components. This parameter reflects the robustness of the graph's structure, revealing how well the individual parts are held together. A graph with elevated toughness is indicative of strong connections, whereas a graph with diminished toughness may exhibit vulnerabilities in its connectivity [16]. The scattering number, in particular, is noted for its representation of a trade-off between the effort to damage a network and the extent of the resulting damage. Specifically, it measures the degree to which a network can be partitioned into smaller, less connected components when subjected to targeted or random attacks [58]. Furthermore, it is also related to other vulnerability parameters, namely toughness and integrity [57]. Another important parameter is tenacity, which is used to measure the vulnerability of networks by taking into account both the effort required to compromise a network and the extent of the damage inflicted [32]. In addition, the rupture degree serves as a significant vulnerability parameter, as it quantifies the level of damage needed to result in the complete breakdown of a network [38].

The primary technical challenge in the evolution of communication technologies is the formulation of a general interconnection network that supports communication between processors. Achieving a balance between high-performance and cost-effective wiring is vital for successful network architecture. Several methodologies can be employed to create an effective interconnection network, such as utilizing finite automata to model the cross-product of these networks [21], graph construction models [19], and with one notable approach being the application of cayley graphs [30]. This technique addresses various performance criteria essential for smooth operation, including large vertex symmetry, minimal degree, small diameter, high connectivity, simple routing issues, and the alleviation of congestion problems [54].

Wrapped butterfly graphs represent a wider category within the realm of butterfly graphs, specifically, they are isomorphic to degree four cayley graphs [53]. Butterfly graphs are known as one of the most popular interconnection networks due to their enormous applications in fields like parallel computing [49] and sorting networks [22]. While wrapped butterfly graphs encompass all the attributes of standard butterfly graphs, they also possess the distinctive feature of large vertex symmetry. This enhancement may contribute to their emergence as a prominent interconnection network, akin to butterfly networks in the future.

In this paper, we analyze the diverse vertex vulnerabilities associated with wrapped butterfly networks, organized in the following manner. The next section will review the concept of vertex vulnerabilities and offer a concise survey of relevant literature, emphasizing the determination of vertex vulnerability within different graph classes along with some basic definitions which are used in this study. Section 3 introduces the networks of focus, i.e., wrapped butterfly network and outlines the mathematical formalities required for our subsequent analysis. Section 4 will assess the various vertex vulnerabilities, specifically integrity, toughness, scattering number, tenacity, and rupture degree. Section 5 discusses the analysis of the different results obtained.

2 Vertex Vulnerability: Integrity, Toughness, Scattering Number, Tenacity, and Rupture Degree

This section contains the fundamental concepts and findings from the literature that we will be referencing in our evidence along with a concise literature survey.

A graph $G = (V, E)$ comprises a set V of vertices (nodes) and a set E of edges (links). The following symbols and definitions will be employed consistently throughout the paper: Consider a simple graph S with a vertex set V and an edge set E. For a subset U $\subseteq$ V, let U, ω(S $-$ U) and m(S$-$U) represent a cut set of S, the number of components, and the order of the largest component of S $-$ U, respectively. A component of a graph is defined as a maximal subgraph in which a path exists from every node to every other node (i.e., they are mutually reachable) and the number of maximal subgraphs are defined as the number of components.

If S $-$ U is disconnected or consists of just one vertex, then a set U $\subset$ V is termed a cut set of G. If each edge in S has at least one end in U, then any subset U$\subset$ V is considered a covering of S. When there is no subset U′ that covers S with $|$U′$|$ $>$ $|$U$|$, then a covering U defined as the minimal covering. Let β(S) denote the number of vertices in a minimal covering of S, which is also known as its covering number. A vertex-independent set is a group of vertices in a graph S that prevents any two of its vertices from being neighbors and the maximum cardinality of such set is defined as independent number and denoted as α(S).

Various parameters have been defined in the literature to quantify the vulnerability of networks, which include the following:

The vertex connectivity, denoted as $\kappa(\mathrm{S})$ is defined as the minimum cardinality among all cut sets of S: $\kappa(\mathrm{S}) = \min\{|\mathrm{U}|\colon \mathrm{U} \subset V(S)\}$. The toughness of a graph S is denoted and defined by,

$$\mathrm{t}(\mathrm{S}) = \min\left\{\frac{|\mathrm{U}|}{\omega(\mathrm{S}-\mathrm{U})} : \mathrm{U} \subset \mathrm{V}(\mathrm{S})\right\}.$$

The scattering number denoted as $\mathrm{sc}(\mathrm{S})$, is defined as,

$$\mathrm{sc}(\mathrm{S}) = \max\{\omega(\mathrm{S}-\mathrm{U}) - |\mathrm{U}|\colon \mathrm{U} \subset \mathrm{V}(\mathrm{S})\}.$$

The rupture degree $\mathrm{r}(\mathrm{S})$ is defined as,

$$\mathrm{r}(\mathrm{S}) = \max\{\omega(\mathrm{S}-\mathrm{U}) - |\mathrm{U}| - m(\mathrm{S}-\mathrm{U})\colon \mathrm{U} \subset \mathrm{V}(\mathrm{S})\}.$$

The integrity $\mathrm{I}(\mathrm{S})$ is given by,

$$\mathrm{I}(\mathrm{S}) = \min\{|\mathrm{U}| + m(\mathrm{S}-\mathrm{U})\colon \mathrm{U} \subset \mathrm{V}(\mathrm{S})\}.$$

The tenacity is defined as,

$$\mathrm{T}(\mathrm{S}) = \min\left\{\frac{(|\mathrm{U}| + m(\mathrm{S}-\mathrm{U}))}{\omega(\mathrm{S}-\mathrm{U})} : \mathrm{U} \subset \mathrm{V}(\mathrm{S})\right\}.$$

The identification of vertex vulnerability parameters is recognized as NP-complete in the context of general graphs. This characteristic prompts a keen interest in analyzing vertex vulnerability parameters within particular classes of graphs, a topic that has been the subject of rigorous investigation. The identification of vulnerable parameters is fundamentally structured around three essential categories:

(a) Cutting.

(b) Covering.

(c) Closeness.

Each category includes parameters that are arranged in a two-dimensional array, which classifies them based on their nature as either vertex or edge parameters, and further distinguishes them as deterministic (related to vulnerability) or probabilistic (associated with reliability) [6]. The present study highlights the parameters of vertex vulnerability associated with cutting, specifically addressing integrity, toughness, scattering number, tenacity, and rupture degree.

In Table 1, we have compiled a few relevant research concerning vertex vulnerability across various graph classes and interconnection networks.

Table 1: Vertex vulnerability of various classes.

References	**Graphs**	**Vertex vulnerability parameters**
Sundareswaran and Swaminathan [50]	Gear Graphs	Integrity
Mamut and Vumar [41]	Kronecker product of complete graphs	Connectivity, integrity, toughness, tenacity and scattering number
Choudum and Priya [15]	Cartesian product of complete graphs and grids	Tenacity
Bagga et al. [7]	Special families of graphs and combination of graphs like- complete graph, null graph, star graph, path graph, cycle graph, Comet graph, complete bipartite graph, and any complete multipartite graph of order p and largest partite set of order r	Integrity
Dündar and Aytaç [20]	Total graphs	Integrity
Basavanagoud et al. [12]	Graph operations and special graphs like: the rooted tree and Kragujevac tree, Mycielskian of G, total closed neighborhood graph, splitting graph, complement of a wheel graph, complement of path graph and complement of cycle graph	Integrity
Vince [52]	Cubic graphs	Upper bound on Integrity
Basavanagoud and Policepatil [11]	Wheel related graphs	Integrity
Basavanagoud et al. [10]	Total transformation graph	Integrity
Atici et al. [3]	Small cage graphs	Integrity
Muhiuddin et al. [45]	$m-$polar fuzzy graphs	Integrity
Vasu et al. [51]	Honeycomb networks	Integrity, toughness, tenacity and scattering number
Basavanagoud et al. [9]	Quasi-total graph of some basic graph families	Integrity
Mamut and Vumar [40]	Middle graphs of some classes of graphs	Integrity
Aytaç [5] / Li et al. [37]	Total graphs of specific families of graphs	Tenacity
Cozzens et al. [18]	Harary graphs	Tenacity
Ma et al. [39]	Torus $P_n \times C_m$	Tenacity
Khoshnood et al. [25]	Generalized Petersen graphs	Tenacity
Kirlangiç [27]	Gear graphs	Rupture degree

References	**Graphs**	**Vertex vulnerability parameters**
Kavitha et al. [24]	Book graph	Tenacity and rupture degree
Li et al. [34]	Permutation graphs of complete bipartite graphs	Tenacity and rupture degree
Aytac and Odabas [4]	Composite graphs	Rupture degree
Aslan and Kirlangi ç[2]	Gear graphs	Scattering number and toughness
Li et al. [38]	Specific classes of graphs	Rupture degree
Paulraja and Sheeba-Agnes [46]	Tensor product of complete equipartite graphs	Toughness, scattering number, integrity and tenacity
Li and Li [33]	Harary graphs	Rupture degree
Li et al. [35]	Harary graphs	Integrity
Kirlangiç and Aytaç [28]	Thorn graphs	Scattering number
Kirlangiç [26]	Binomial tree	Scattering number
Broersma et al. [13]	Interval graph	Scattering number
Chen and Zhang [14]	Bicyclic graphs	Scattering number
Kratsch et al. [29]	Interval graph, circular arc-graphs, permutation graphs, circular permutation graphs, trapezoid graphs and co-comparability graphs of bounded dimension	Scattering number and toughness
Kaval and Kirlangiç [23]	$K_{1,m} \times K_{1,n}$, $K_{1,m} \times P_n$, $K_{1,m} \times C_n$ and $K_2 \times C_n$	Scattering number
Markenzon and Waga [42]	Strictly chordal graphs	Scattering number
Rajasingh et al. [48]	Split graphs, regular caterpillars and a class of meshes	Rupture degree
Agnes and Geethanjaliyadav [1]	Some classes of graphs	Rupture degree
Moazzami and Vahdat [44]	Interval graph	Tenacity and rupture degree
Ye [56]	Power and total graphs	Integrity, toughness, rupture degree
Li [36]	Trees	Rupture degree

In summary, we would like to reiterate the following result that will be essential for our subsequent discussions.

Theorem 2.1. [58] *Let S be an n-order, non-complete connected graph. Then,*

$$2 - \kappa(S) \leq sc(S) \leq n - 2\kappa(S).$$

Theorem 2.2. [7] *Let S be an n-order, non-complete connected graph. Then,*

$$\delta(S) + 1 \leq I(S) \leq \beta(S) + 1.$$

Theorem 2.3. [38] *Let S be an n-order, non-complete connected graph. Then,*

$$3-n \leq r(S) \leq n-3.$$

Theorem 2.4. [16] *Let S be an n-order graph. Then,*

$$\frac{\kappa(S)}{\alpha(S)} \leq t(S) \leq \frac{\kappa(S)}{2}.$$

Theorem 2.5. [43] *Let S be an n-order graph. Then,*

$$\frac{\kappa(S+1)}{\alpha(S)} \leq t(S) \leq \frac{n-\kappa(S+1)}{\alpha(S)}.$$

Lemma 2.1. [57] *Let S be an n-order graph. Then,*

$$t(S) \geq \frac{\kappa(S)}{\kappa(S)+sc(S)}.$$

3 Wrapped Butterfly Graph

WBF(n), $n \geq 3$, the $n-$dimensional wrapped butterfly network, consists of n rows, or levels, with 2^n vertices or columns per row. A vertex in WBF(n) is denoted by,

$$\{(v;i)/v=(v_1,v_2,...v_n),\quad v_i=0 \text{ or } 1,\quad 1\leq i\leq n\}.$$

In WBF(n), an edge connects two vertices $(v;i)$ and $(u;j)$ if and only if $j=i+1(\mod n)$ and either (i) $v=u$ or (ii) v varies from u in exactly the j^{th} bit. The $n-$dimensional wrapped butterfly network is a particular kind of butterfly network BF(n) in which the initial and final layers of BF(n) are integrated into a single level. That is, vertex $(v;1)$ merges into vertex $(v;n)$ for each v [54]. The $n-$level wrapped butterfly graph WBF(n) has $n2^n$ vertices and $2^{n+1}n$ edges and is a constant $4-$degree Cayley graph. The $3-$dimensional wrapped butterfly network is depicted in Figure 1, [17].

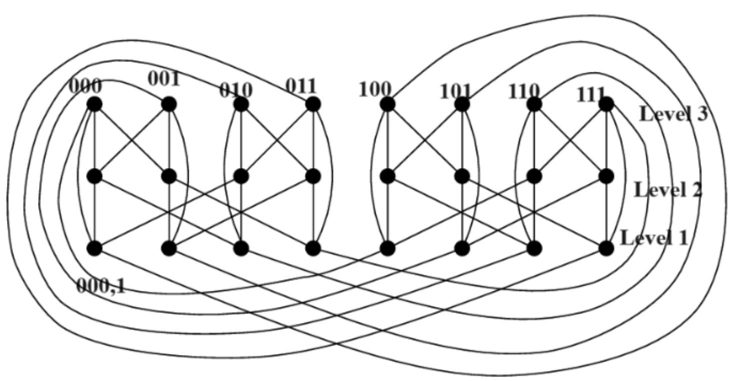

Figure 1: The $3-$dimensional wrapped butterfly network.

We will utilize the subsequent lemmas, proposition, and corollary as an essential part in proving our main theorem.

Proposition 3.1. *Let n be an integer with $n \geq 3$, then,*

(a) $\alpha(WBF(n)) = \left\lfloor \frac{n}{2} \right\rfloor 2^n$.

(b) $\beta(WBF(n)) = \left\lceil \frac{n}{2} \right\rceil 2^n$.

Proof.

(a) Case(i): **When n is even**.
We start selecting all the vertices positioned at Level 1, i.e., 2^n number of vertices. As no two vertices are connected in the same row, we continue selecting all the vertices from the alternative levels upto level $(n-1)$; hence, the total number of vertices selected is $\frac{n}{2}2^n$.

Case(ii): **When n is odd**.
We proceed by selecting all the vertices positioned at Level 1 and continue selecting all the vertices from the alternative levels upto level $(n-2)$. Hence, the total number of non-adjacent vertices selected are $\left\lfloor \frac{n}{2} \right\rfloor 2^n$.

(b) Case(i): **When n is even**.
Let U be a covering set. To cover all the edges of row 1 and row 2 we choose $|\mathrm{U}| = 2^n$ from row 1. Now to cover the edges of rows 2 and 3, we can choose $|U| = 2^n$ from row 3 Continuing, we choose vertices from alternative rows (odd level) until one end vertex of every edge belongs to the set $|U|$. Hence, $|\mathrm{U}| = \frac{n}{2}2^n = \left\lceil \frac{n}{2} \right\rceil 2^n$.

Case(ii): **When n is odd**.
We proceed the same way by selecting $|\mathrm{U}| = 2^n$ from row 1 and continuing, we choose vertices from alternative rows (odd level). To cover the edges of row $(n-1)$ and row n, we choose $|\mathrm{U}| = 2^n$ from row n and hence, $|\mathrm{U}| = \left\lceil \frac{n}{2} \right\rceil 2^n$.

□

Lemma 3.1. *Let U be a cut set of $S = WBF(n)$, where n is an integer with $n \geq 3$.*

(a) $|U| = \omega(S-U) = n2^{(n-1)}$, *and* $m(S-U) = 1$, *when n is even.*

(b) *Let $\omega(S-U) = 2$ and let S_1 and S_2 denote the components of $S-U$ then if either $|S_1|$ or $|S_2| = 1$, then $|U| = 4$, else $|U| > 4$.*

Proof.

(a) We choose $|\mathrm{U}| = 2^n$ from level n, the graph becomes disconnected, and $\omega(\mathrm{S}-\mathrm{U}) = 2$. Additionally, $|\mathrm{S}_1| = |\mathrm{S}_2|$. We choose 2^n from alternate rows up to Level 2. This results in an equal number of rows containing isolated vertices as the number of vertices selected to make the graph disconnected. Hence, when $|\mathrm{U}| = n2^{(n-1)}$, then $\omega(\mathrm{S}-\mathrm{U}) = n2^{(n-1)}$, see Figure 2 (for simplicity, we avoid the edges that wrap Level 1 and Level n), where the red color vertices denote the vertex cut $|\mathrm{U}|$ and black color vertices denote the number of components.

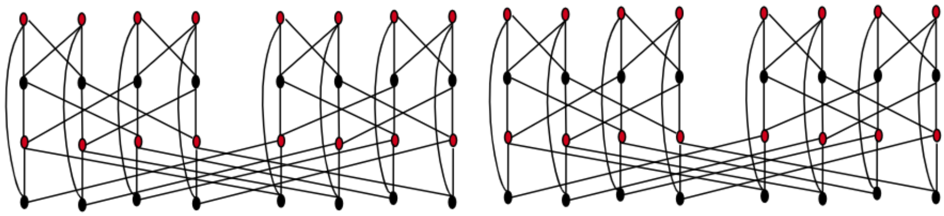

Figure 2: ω(S − U) and |U| for 4−dimensional wrapped butterfly network.

(b) We know that κ(S) $= 4$. When |U| $= 4$, then ω(S − U) $= 2$, and we are finished with the proof if either |S$_1$| or |S$_2$| $= 1$. Let's now assume, maintaining generality, that |S$_1$| ≥ 3 and |S$_2$| ≥ 2. Now, by the definition of WBF(n), we know that no two vertices have the same set of adjacent vertices. Hence, when κ(S) $= 4$, then |U| > 4, which is a contradiction to our assumption that |U| $= 4$. Hence, if ω(S−U) $= 2$ and |S$_1$| or |S$_2$| $= 1$, then |U| $= 4$ else |U| > 4.

□

Lemma 3.2. *$\omega(S-U) = 4$ when $|U| = 2^n$, where U is a collection of vertices selected from Level 1.*

Proof. Since the graph WBF(n) is symmetric and regular, by definition of the topology of the graph, the edge connects two vertices $(v; i)$ and $(u; j)$ if and only if $j = i + 1(\mod n)$ and either (i) v = u or (ii) v varies from u in exactly the j^{th} bit. In WBF(n), when selecting |U| $= 2^n$ from Level 1, the graph S − U becomes disconnected, and the number of components becomes 4. Hence, ω(S − U) $= 4$. □

Corollary 3.1. *When $|U| = 2^n$, where U represents a set of vertices chosen either from Level 1 or Level n, the order of each component in $(S-U)$ is uniform.*

Proof. In WBF(n) with |U| $= 2^n$, if U is selected from Level 1, there are 4 components, each having an order of $\frac{2^n}{2^2}(n-1)$. On the other hand, when U is chosen from level n, the number of components decreases to 2, and the order of each component is $\frac{2^n}{2}(n-1)$. Thus, the order of each component in (S − U) is consistent, regardless of whether it is derived from Level 1 or Level 4. □

Lemma 3.3. *Given a cut set U, let S = WBF(n) with $n \geq 3$ and n being odd then,*

$$\omega(S-U) \leq (n-1)2^{(n-1)}.$$

Proof. Let |U| $= x$. If U is a covering set, i.e., |U| $= \left\lceil \frac{n}{2} \right\rceil 2^n$, then it's simple to confirm that ω(S − U) $= (n-1)2^{(n-1)}$ as $\alpha(S) + \beta(S) = n$ where n is the order of the graph S. Now, suppose $x > \left\lceil \frac{n}{2} \right\rceil 2^n$, then we have ω(S − U) $\leq n2^n - x = n2^n - \left\lceil \frac{n}{2} \right\rceil 2^n = (n-1)2^{(n-1)}$. When $x < \left\lceil \frac{n}{2} \right\rceil 2^n$, then as it is obvious that ω(S − U) $\leq \alpha$(S) $= (n-1)2^{(n-1)}$. □

Lemma 3.4. *$\omega(S-U) = (n-1)2^{(n-1)}$, when $|U| = n2^{(n-1)}$ or $(n+1)2^{(n-1)}$ where U is a cut set for $n \geq 3$ and n is odd.*

Proof. When $|\text{U}| = \beta(\text{S})$, it is evident that $\omega(\text{S}-\text{U}) \leq \alpha(\text{S})$ and $m(\text{S}-\text{U}) \geq 1$. Specifically, when $|\text{U}| = (n+1)2^{(n-1)}$, it results in $\omega(\text{S}-\text{U}) = (n-1)2^{(n-1)}$ by Lemma 3.3. S consists of n rows, each containing vertices representing columns. The columns are partitioned into two halves: the first set of vertices $\left(\frac{n2^n}{2}\right)$ represents one half, while the next set of vertices $\left(\frac{n2^n}{2}\right)$ represents the other half. Now, without loss of generality, we choose $|\text{U}| = \frac{2^n}{2}$ from the n^{th} row of the first half and proceed by selecting $\frac{2^n}{2}$ number of vertices from alternative rows. Therefore, the total number of vertices selected from the first half is $\frac{(n+1)}{2}2^{(n-1)}$. It is evident that when $|\text{U}| = \frac{(n+1)}{2}2^{(n-1)}$, $\omega(\text{S}-\text{U}) = \frac{(n-1)}{2}2^{(n-1)}$.

Similarly, we choose $|\text{U}| = \frac{2^n}{2}$ from the second row of the second half and proceed by selecting $\frac{2^n}{2}$ number of vertices from alternative rows; thus, the total number of vertices chosen from the second half is $\frac{(n-1)}{2}2^{(n-1)}$.

It is easy to see that when $|\text{U}| = \frac{(n-1)}{2}2^{(n-1)}$, $\omega(\text{S}-\text{U}) = \frac{(n-1)}{2}2^{(n-1)}$. The second set of vertices in row 1 and the second set of vertices in row n are connected by an edge, resulting in $m(\text{S}-\text{U}) = 2$. Hence, the total number of vertices selected from both the first and second halves is $\frac{(n+1)}{2}2^{(n-1)} + \frac{(n-1)}{2}2^{(n-1)} = n2^{(n-1)}$ vertices and

$$\omega(\text{S}-\text{U}) = \frac{(n-1)}{2}2^{(n-1)} + \frac{(n-1)}{2}2^{(n-1)} = (n-1)2^{(n-1)},$$

see Figure 3(a) (for simplicity, we avoid the edges that wrap Level 1 and Level n).

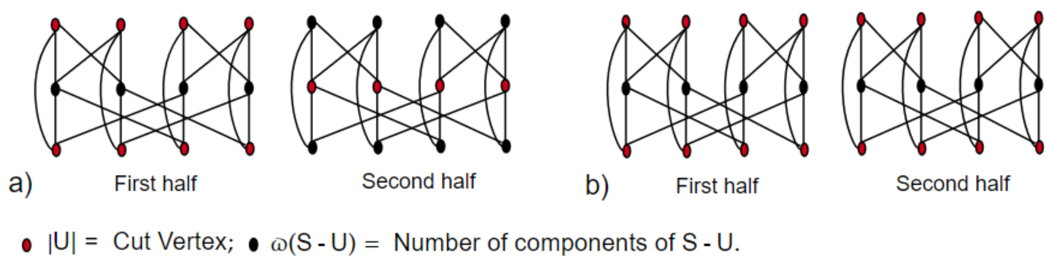

Figure 3: 3-dimensional wrapped butterfly network:
(a) $\omega(\text{S}-\text{U}) = 8$ and $|\text{U}| = 12$.
(b) $\omega(\text{S}-\text{U}) = 8$ and $|\text{U}| = 16$.

Similarly, when $|\text{U}| = (n+1)2^{(n-1)}$, then its easy to verify that $\omega(\text{S}-\text{U}) = (n-1)2^{(n-1)}$, see Figure 3(b). $\square$

4 Vertex Vulnerability Parameters for Wrapped Butterfly Graphs

The key outcome of our paper is presented as follows:

Theorem 4.1. $I(WBF(n)) = 2^{(n-2)}[3+n]$, $n \geq 3$.

Proof. Our main aim is to choose $|\mathrm{U}|$ very small in such a way that $m(\mathrm{S}-\mathrm{U})$ is also small. We know that $\kappa(\mathrm{WBF}(n)) = 4$; then, by Lemma 3.1(b), for $|\mathrm{U}| = 4$, it follows that $\omega(\mathrm{S}-\mathrm{U}) = 2$ and $m(\mathrm{S}-\mathrm{U}) = n2^n - 5$. Hence, we choose $|\mathrm{U}| > 4$. Now, by Corollary 3.1, when $|\mathrm{U}|=2^n$ (choosing from the first level alone), $m(\mathrm{S}-\mathrm{U}) = \dfrac{2^n}{2^2}(n-1)$.

Therefore, $|\mathrm{U}| + m(\mathrm{S}-\mathrm{U}) = 2^n + \dfrac{2^n}{2^2}(n-1) = 2^{n-2}[n+3]$. By definition,

$$\mathrm{I(S)} = \min(|\mathrm{U}| + m(\mathrm{S}-\mathrm{U})) \leq 2^{n-2}[n+3].$$

When $|\mathrm{U}| = 2^n$, it is observed that $\omega(\mathrm{S}-\mathrm{U}) = 4$ by Lemma 3.2. As $m(\mathrm{S}-\mathrm{U}) \geq \dfrac{n-|\mathrm{U}|}{\omega(\mathrm{S}-\mathrm{U})}$, we get $m(\mathrm{S}-\mathrm{U}) \geq 2^{n-2}[n-1]$. Therefore,

$$\mathrm{I(S)} = \min(|\mathrm{U}| + m(\mathrm{S}-\mathrm{U})) \geq 2^n + 2^{n-2}[n-1] = 2^{n-2}[n+3],$$

consequently, we deduce that $\mathrm{I}(\mathrm{WBF}(n)) = 2^{(n-2)}[3+n]$. □

Theorem 4.2. *Let $S = WBF(n)$, $n > 2$. Then,*

(1) $t(S) = \begin{cases} 1, & n \text{ is even}, \\ \dfrac{n}{(n-1)}, & n \text{ is odd}. \end{cases}$

(2) $sc(S) = \begin{cases} -2, & n \text{ is odd}, \\ 0, & n \text{ is even}. \end{cases}$

(3) $T(S) = \begin{cases} \dfrac{1+n2^{(n-1)}}{n2^{(n-1)}}, & n \text{ is even}, \\ \dfrac{n2^{(n-1)}+2}{(n-1)2^{(n-1)}}, & n \text{ is odd}. \end{cases}$

(4) $r(S) = \begin{cases} -2^{(n-1)} - 2, & n \text{ is odd}, \\ -1, & n \text{ is even}. \end{cases}$

Proof.

(1) **When n is even.**
Let T_1 be a cut set. When $|T_1| = n2^{(n-1)}$, then by Lemma 3.1(a), $\omega(\mathrm{S}-T_1) = n2^{(n-1)}$, thus $\dfrac{|T_1|}{\omega(\mathrm{S}-T_1)} = 1$, which implies that $\mathrm{t(S)} \leq 1$. In a WBG(S), the set $|\mathrm{U}| = n2^{(n-1)}$ is the unique cut set such that $\omega(\mathrm{S}-\mathrm{U}) = |\mathrm{U}|$, as when n is even $\alpha(\mathrm{S}) = \beta(\mathrm{S})$, and for any other cut set, it is guaranteed that $\omega(\mathrm{S}-\mathrm{U}) \leq |\mathrm{U}|$. Thus, the minimum value of $\dfrac{|\mathrm{U}|}{\omega(\mathrm{S}-\mathrm{U})}$ is achieved when $\omega(\mathrm{S}-\mathrm{U}) = |\mathrm{U}|$, hence,

$$\mathrm{t(S)} \geq \min\left(\frac{|\mathrm{U}|}{\omega(\mathrm{S}-\mathrm{U})}\right) = \left\lceil\frac{n}{2}\right\rceil 2^n / \left\lfloor\frac{n}{2}\right\rfloor 2^n = 1,$$

hence the result.

When n is odd.
Let T_1 be a cut set such that when $|T_1| = n2^{(n-1)}, \omega(\mathrm{S} - T_1) = (n-1)2^{(n-1)}$, by Lemma 3.4, hence $\dfrac{|T_1|}{\omega(\mathrm{S} - T_1)} = \dfrac{n2^{(n-1)}}{(n-1)2^{(n-1)}}$, we have $\mathrm{t(S)} \leq \dfrac{n2^{(n-1)}}{(n-1)2^{(n-1)}}$.

According to Lemma 3.4, $\omega(\mathrm{S}-\mathrm{U}) \leq (n-1)2^{(n-1)}$ holds true for $n2^{(n-1)} \leq \mathrm{U} \leq (n+1)2^{(n-1)}$. The minimum value of $\dfrac{|\mathrm{U}|}{\omega(\mathrm{S}-\mathrm{U})}$ is achieved when $|\mathrm{U}| = n2^{(n-1)}$ and $\omega(\mathrm{S}-\mathrm{U}) = (n-1)2^{(n-1)}$. Therefore, it follows that,

$$\mathrm{t}(S) \geq \frac{n2^{(n-1)}}{(n-1)2^{(n-1)}} = \frac{n}{(n-1)}.$$

(2) **When n is even.**
Let $|T_1|$ be a cut set such that $|T_1| = \dfrac{n}{2}2^n$. In this case, when $|T_1| = \dfrac{n}{2}2^n$, it implies $\omega(\mathrm{S} - T_1) = \dfrac{n}{2}2^n$, leading to $\mathrm{sc(S)} \leq 0$. Conversely, by Lemma 2.1, we know that $\dfrac{\kappa(\mathrm{S})}{\kappa(\mathrm{S}) + \mathrm{sc(S)}} \leq 1$ as $\mathrm{t(S)} = 1$. Thus, it follows that $\mathrm{sc(S)} \geq 0$. Consequently, $\mathrm{sc(S)} = 0$.

When n is odd.
If $\omega(\mathrm{S}-\mathrm{U}) = 2$, then we know that by Lemma 3.1(b) $|\mathrm{U}| = 4$. Hence,

$$\min(\omega(\mathrm{S}-\mathrm{U})) - |\mathrm{U}| \leq 2 - 4 = -2.$$

Now, by Theorem 2.1(i), $2 - \kappa(\mathrm{S}) \leq \mathrm{sc(S)} = 2 - 4 \leq \mathrm{sc(S)} = \mathrm{sc(S)} \geq -2$. Hence, $\mathrm{sc(S)} = -2$.

(3) **When *n* is even.**
Let U be a cut set of S with $|\mathrm{U}| = x$. The remaining graph $\mathrm{S}-\mathrm{U}$ has at most x components, and therefore, $m(\mathrm{S}-\mathrm{U}) \geq \dfrac{n2^n - x}{x}$. Since $m(\mathrm{S}-\mathrm{U}) \geq \dfrac{n2^n - x}{x} \geq 1$, x must be at most $n2^{(n-1)}$. Consequently, we obtain $T(\mathrm{S}) \geq \min\left(\dfrac{\frac{n2^n - x}{x} + x}{x}\right)$, where $x \leq n2^{(n-1)}$.

Now, let's analyze the function $f(x) = \dfrac{\frac{n2^n - x}{x} + x}{x}$. It's evident that $f'(x) = \dfrac{x - n2^{n+1}}{x^3}$. Since $x \leq n2^{(n-1)}$, we have $f'(x) < 0$, implying that $f(x)$ is a decreasing function. Therefore, the minimum value of $f(x)$ occurs at $x = n2^{(n-1)}$, and $f_{\min}(x) = \dfrac{1 + n2^{(n-1)}}{n2^{(n-1)}}$. Hence, $T(\mathrm{S}) \geq \dfrac{1 + n2^{(n-1)}}{n2^{(n-1)}}$.
Alternatively, considering U′ as the covering set of S, where $|\mathrm{U}'| = n2^{(n-1)}$, $m(\mathrm{S}-\mathrm{U}') = 1$, and $\omega(\mathrm{S}-\mathrm{U}') = n2^{(n-1)}$, we can utilize the definition of tenacity to derive,

$$\mathrm{T(S)} \leq \frac{|\mathrm{S}'| + m(\mathrm{S}-\mathrm{U}')}{\omega(\mathrm{S}-\mathrm{U}')} = \frac{1 + n2^{(n-1)}}{n2^{(n-1)}}.$$

Consequently, we conclude that, $\mathrm{T(S)} = \dfrac{1+n2^{(n-1)}}{n2^{(n-1)}}$.

When *n* is odd.
Referring to Lemma 3.3 and Lemma 3.4, when $|\mathrm{U}| \geq n2^{(n-1)}$, it follows that,

$$\omega(\mathrm{S}-\mathrm{U}) \leq (n-1)2^{(n-1)}, \quad \text{and} \quad m(\mathrm{S}-\mathrm{U}) \geq 1.$$

The minimum value for tenacity occurs when $m(\mathrm{S}-\mathrm{U}) = 2$. Therefore, we have

$$\mathrm{T(S)} \geq \frac{|\mathrm{U}| + m(\mathrm{S}-\mathrm{U})}{\omega(\mathrm{S}-\mathrm{U})} = \frac{n2^{(n-1)}+2}{(n-1)2^{(n-1)}}.$$

Alternatively, let U′ denote the covering set of S, where $|\mathrm{U}'| = n2^{(n-1)}$, $m(\mathrm{S}-\mathrm{U}') = 2$, and $\omega(\mathrm{S}-\mathrm{U}') = (n-1)2^{(n-1)}$.

By the definition of tenacity, we can deduce that, $\mathrm{T(S)} \leq \dfrac{|\mathrm{S}'| + m(\mathrm{S}-\mathrm{U}')}{\omega(\mathrm{S}-\mathrm{U}')} = \dfrac{2+n2^{(n-1)}}{(n-1)2^{(n-1)}}$.

Therefore, it follows that, $T(\mathrm{S}) = \dfrac{2+n2^{(n-1)}}{(n-1)2^{(n-1)}}$.

(4) **When n is even.**
It is easy to see that by Lemma 3.1(a) there is a vertex cut T_1 such that when $|T_1| = \left(\frac{n}{2}\right)2^n$, then $\omega(\mathrm{S}-T_1) = \left(\frac{n}{2}\right)2^n$ and $m(\mathrm{S}-T_1) = 1$ from the definition of rupture degree, we have

$$\mathrm{r(S)} \geq \omega(\mathrm{S}-T_1) - |T_1| - m(\mathrm{S}-T_1) = -1.$$

Let X be an arbitrary vertex cut of G, and set $|\mathrm{X}| = x$. If $x \leq \left(\frac{n}{2}\right)2^n$, then $\omega(\mathrm{S}-\mathrm{X}) \leq x$. Therefore, we have $m(\mathrm{S}-\mathrm{X}) \geq \left\lceil\left(\dfrac{n2^n-x}{x}\right)\right\rceil$. Hence,

$$\omega(\mathrm{S}-\mathrm{X}) - |\mathrm{X}| - m(\mathrm{S}-\mathrm{X}) \leq -\left\lceil\left(\frac{n2^n-x}{x}\right)\right\rceil \leq -1.$$

If $x > \left(\frac{n}{2}\right)2^n$, then $\omega(\mathrm{S}-\mathrm{X}) \leq n2^n - x$. Hence,

$$\omega(\mathrm{S}-\mathrm{X}) - |\mathrm{X}| - m(\mathrm{S}-\mathrm{X}) \leq n2^n - 2x - 1 \leq -1.$$

From the choice of X and the definition of rupture degree, we obtain $\mathrm{r(S)} \leq -1$.

When n is odd.
Let U be an arbitrary cut set of the graph S and set $|\mathrm{U}| = x$. If $n2^{(n-1)} \leq x \leq (n+1)2^{(n-1)}$, then by Lemma 3.4, $\omega(\mathrm{S}-\mathrm{U}) = (n-1)2^{(n-1)}$. When $n2^{(n-1)} \leq x < (n+1)2^{(n-1)}$, we have $m(\mathrm{S}-\mathrm{U}) = 2$ and when $x = (n+1)2^{(n-1)}$, then $m(\mathrm{S}-\mathrm{U}) = 1$. Hence,

$$\omega(\mathrm{S}-\mathrm{U}) - |\mathrm{U}| - m(\mathrm{S}-\mathrm{U}) = (n-1)2^{(n-1)} - x - m(\mathrm{S}-\mathrm{U}).$$

It is easy to verify that the maximum value of $(n-1)2^{(n-1)} - x - m(\mathrm{S}-\mathrm{U})$ is obtained when $x = n2^{(n-1)}$ and $m(\mathrm{S}-\mathrm{U}) = 2$, and hence,

$$\begin{aligned}\mathrm{r(S)} &= \max\{\omega(\mathrm{S}-\mathrm{U}) - |\mathrm{U}| - m(\mathrm{S}-\mathrm{U})\}\\ &= (n-1)2^{(n-1)} - n2^{(n-1)} - 2\\ &= -2^{(n-1)} - 2.\end{aligned}$$

When $x > (n+1)2^{(n-1)}$, we have $\omega(\mathrm{S}-\mathrm{U}) \leq n2^n - x$, and $m(\mathrm{S}-\mathrm{U}) \geq 1$. Now,

$$\begin{aligned}\omega(\mathrm{S}-\mathrm{U}) - |\mathrm{U}| - m(\mathrm{S}-\mathrm{U}) \leq n2^n - 2x - 1 &= n2^n - 2(n+1)2^{(n-1)} - 1\\ &= n2^n - (n+1)2^n - 1\\ &= -2^n - 1\\ &= -2^{(n-1)}2 - 2 + 1\\ &< -2^{(n-1)} - 2,\end{aligned}$$

Hence, $\mathrm{r}(\mathrm{S}) \leq -2^{(n-1)} - 2$.

When $x < n2^{(n-1)}$, then $\omega(\mathrm{S}-\mathrm{U}) \leq (n-1)2^{(n-1)} = x - 2^{(n-1)}$ and $m(\mathrm{S}-\mathrm{X}) \geq 2$. Hence,

$$\omega(\mathrm{S}-\mathrm{U}) - |\mathrm{U}| - m(\mathrm{S}-\mathrm{U}) \leq -2^{(n-1)} - 2, \quad \text{implies} \quad \mathrm{r}(\mathrm{S}) \leq -2^{(n-1)} - 2.$$

It is easy to see that by Lemma 3.4, there is a vertex cut T_1 such that when $|T_1| = n2^{(n-1)}$ then $\omega(\mathrm{S}-T_1) = (n-1)2^{(n-1)}$ and $m(\mathrm{S}-T_1) = 2$. We obtain

$$\mathrm{r}(\mathrm{S}) \geq \omega(\mathrm{S}-T_1) - |T_1| - m(\mathrm{S}-T_1) = -2^{(n-1)} - 2,$$

from the rupture degree definition. Hence, the proof.

□

Algorithm 1: Integrity Assignment

Input: A wrapped butterfly graph WBF(n)
Output: Integrity of WBF(n)

/* Finding the optimal cut set */
Initialize U $\leftarrow \emptyset$;
for $i \leftarrow 1$ ***to*** n **do**
  if $i = 1$ **then**
    U $\leftarrow \{v, i\}$;
    $|\mathrm{U}| \leftarrow \mathrm{len}(\mathrm{U})$;
/* order of the largest connected component after removing U from S */
S_$removed$_U $\leftarrow$ S.copy();
S_$removed$_U.remove_nodes_from(U);
largest_cc $\leftarrow$ max(nx.connected_components(S_$removed$_U), key=len);
m_S_$minus$_U $\leftarrow$ len($largest\_cc$);

Initially, Algorithm 1 begins by initializing an empty set U , which takes $O(1)$ time. It then iterates from $i = 1$ to $i = n$, where only on the first iteration, vertices from Level 1 are chosen to the set U . Although this loop theoretically runs $O(n)$ times, the actual operations within the `if` condition only execute once, making this part effectively $O(1)$.

Following this, the algorithm creates a copy of the graph S, which takes $O(\mathrm{V}+\mathrm{E})$ time for a graph with V vertices and E edges, and removes the nodes in U from the copied graph. The removal of nodes has a complexity of $O(|\mathrm{U}| + d(\mathrm{U}))$, where $d(\mathrm{U})$ represents the sum of the degrees of nodes in U.

Next, the algorithm finds the largest connected component after the removal, which involves identifying connected components in the graph. This operation is $O(\mathrm{V}+\mathrm{E})$, as it requires traversing the graph. Thus, the overall complexity of the algorithm combines these steps into,

$$O(\mathrm{V}+\mathrm{E}+|\mathrm{U}|+d(\mathrm{U})+n).$$

However, since n, $|\mathrm{U}|$, and $d(\mathrm{U})$ are typically small relative to V and E in large graphs, the algorithm's complexity is effectively $O(\mathrm{V}+\mathrm{E})$. This makes it efficient for analyzing the integrity of large graphs by examining connected components after specific node removals.

Algorithm 2: Toughness, Tenacity and Rupture degree Assignment

Input: A wrapped butterfly graph S = WBF(n)
Output: Toughness, Tenacity and Rupture degree of S

if $n\%2 == 0$ **then**
  U $\leftarrow$ {vertex | vertex in levels_i and level_i$\%2 == 1$};
else
  for *each level_i in WBF*(n) **do**
    total_vertices_in_each_level_i $\leftarrow 2^n$;
    first_half $\leftarrow$ first $(2^n)/2$ vertices from level_i;
    second_half $\leftarrow$ remaining $(2^n)/2$ vertices from level_i;
    From first_half, choose all the alternative vertices from level_n to level_1;
    From second_half, choose all the alternative vertices from level_(n-1) to level_2;
  U $\leftarrow$ selected vertices from the previous steps;
/* order of the largest connected component after removing U from S */
S_$removed$_U $\leftarrow$ S.copy();
S_$removed$_U.remove_nodes_from(U);
largest_cc $\leftarrow$ max(nx.connected_components(S_$removed$_U), key=len);
m_S_$minus$_U $\leftarrow$ len($largest\_cc$);
/* number of components in the graph after removing U from S */
w_S_$minus$_U $\leftarrow$ len(list(nx.connected_components(S_$removed$_U)));

In order to compute the scattering number associated with WBF(n), the selection of the set U is straightforward and based on Theorem 2.1. For odd values of n, the appropriate choice is U $= \kappa(\mathrm{WBF}(n))$. In contrast, for even values of n, the selection process for U is the same as that used for other parameters when n is even. Therefore, we present Algorithm 2, which facilitates the calculation of toughness, tenacity, and rupture degree. The complexity analysis can be broken down as follows:

First, depending on whether n is even or odd, the algorithm selects a subset of vertices U. If n is even, the subset U is chosen from vertices in odd-numbered levels, requiring $O(\mathrm{V})$ time, where V represents the number of vertices in the graph. When n is odd, the algorithm iterates through each level i in the graph and divides the vertices of each level into two halves, from which it selects alternating vertices. Given that each level has 2^n vertices, and this process must be repeated for each of the n levels, the time complexity for this selection process is $O(n \cdot 2^n)$.

After defining U, the algorithm creates a copy of the graph S, which takes $O(\mathrm{V}+\mathrm{E})$, where E is the number of edges. The selected vertices in U are then removed from the copied graph, an operation that has complexity $O(|\mathrm{U}|+d(\mathrm{U}))$, where $d(\mathrm{U})$ represents the sum of degrees of nodes in U. Next, the algorithm calculates the largest connected component in the modified graph S_$removed$_U by identifying connected components and selecting the one with the largest size. Finding connected components has a complexity of $O(\mathrm{V} + \mathrm{E})$, as it involves traversing the graph.

Finally, the algorithm determines the number of components in the graph after the removal of U by calling a connected components function again, which has a complexity of $O(\mathrm{V} + \mathrm{E})$. Summing these parts, the total complexity is approximately,

$$O(n \cdot 2^n + \mathrm{V} + \mathrm{E} + |\mathrm{U}| + d(\mathrm{U})).$$

Since $n \cdot 2^n$, $|\mathrm{U}|$, and $d(\mathrm{U})$ are typically smaller than V + E in large graphs, the overall complexity can be simplified to

$$O(\mathrm{V} + \mathrm{E}).$$

This makes the algorithm efficient for calculating toughness, tenacity, and rupture degree in large wrapped butterfly graphs.

4.1 Simulation experiment

While the majority of articles rely on theoretical frameworks for their results, our work includes an algorithm and a simulation experiment to enhance comprehension of the efficiency of WBF(n) within a network context. Hence in this section, we delve into the planning, implementation, and results of the simulation experiment to assess the performance of Algorithms 1 and 2. Initially, we identify all potential combinations of cut sets U within a wrapped butterfly graph, subsequently determining $\omega(\mathrm{S} - \mathrm{U})$ and $m(\mathrm{S} - \mathrm{U})$. The outcomes are then evaluated for various properties, including integrity, toughness, scattering number, tenacity, and rupture degree pertaining to graph S. In the case where n is even, the topology of the wrapped butterfly graph yields a cut set whose cardinality corresponds to the number of components. This configuration leads to the attainment of the optimal solution. Conversely, when n is odd, various combinations are evaluated to determine the optimal solution.

In the following Table 2, the feasible solutions have been calculated for values of n equal to 3, 4 and 5, from which the optimal solution has been identified.The application of Theorems 2.1, 2.2, 2.3, 2.4, and 2.5 allows us to exclude all solutions that do not conform to the established bounds. The selection process involved choosing the minimum values for integrity, toughness, and tenacity, while the maximum values were selected for rupture degree and scattering number, consistent with their definitions. The findings in Table 2 indicates that the optimal solution aligns with the results generated by Algorithms 1 and 2, thereby demonstrating the algorithms' effectiveness.

Table 2: Comparison of various values of vertex vulnerability parameters generated by Algorithms 1 and 2 with all feasible solutions.

n	Vertex vulnerability parameters	Values	
		All feasible solutions	Optimal solution
3	Integrity	$12, 13, 14, 15, 16$	12
	Toughness	$1.5, 1.6, 1.7, 1.8, 1.9, \ldots, 2$	1.5
	Tenacity	$1.8, 1.9, \ldots, 2.1$	1.8
	Rupture Degree	$-6, -7, -8, -9, \ldots, -21$	-6
	Scattering Number	-2	-2
4	Integrity	$28, 29, 30, 31, 32, 33$	28
	Toughness	$1, 1.6, 1.7, 1.8, \ldots, 2$	1
	Tenacity	1.03	1.03
	Rupture Degree	$-1, -3, -4, -5, \ldots, -61$	-1
	Scattering Number	$-2, 0$	0
5	Integrity	$64, 65, 66, 67, 68, \ldots, 97$	64
	Toughness	$1.25, 1.3, 1.4, 1.5, \ldots, 2$	1.25
	Tenacity	$1.28, 1.3, \ldots, 1.5$	1.28
	Rupture Degree	$-34, -35, -36, -37, \ldots, -157$	-34
	Scattering Number	-2	-2

5 Analysis of The Results

The physical insight into the results presented in this study related to the wrapped butterfly graph WBF(n) reveals key characteristics related to its vulnerability and connectivity. Hence in this section, we give a complete analysis of the results obtained.

The integrity of the graph, represented by $\mathrm{I}(\mathrm{WBF}(n)) = 2^{(n-2)}[3+n]$, demonstrates that the graph's robustness increases exponentially with n, indicating that larger graphs are more robust against node failures.

The toughness of the graph, which varies depending on whether n is even or odd, shows that for even values of n, the graph exhibits a toughness value of 1, which implies that it requires removing a substantial portion of the graph to disconnect it. However, the minimal ratio $\dfrac{|\mathrm{U}|}{\omega(\mathrm{S} - \mathrm{U})} = 1$ means that this is the most efficient way to disconnect the graph and reflects that the graph is resilient to fragmentation, but once the right cut set U is found, it will disconnect in an optimal manner. For odd values of n, the toughness is given by $\dfrac{n}{n-1}$, which is slightly greater than 1, signaling that it is still quite resistant to disconnection and may require slightly more effort to break connectivity. The scattering number, which reflects the extent of fragmentation when vertices are removed, is negative for odd n i.e., -2, and zero for even n, highlighting a stronger resistance to fragmentation in odd configurations.

The tenacity of the graph, which measures the minimum number of nodes whose removal disconnects the graph, shows distinct patterns based on n's parity. For even n, the tenacity formula $\dfrac{1+n2^{(n-1)}}{n2^{(n-1)}}$ reflects a higher degree of robustness to disconnection, while for odd n, $\dfrac{n2^{(n-1)}+2}{(n-1)2^{(n-1)}}$ implies that a slightly greater number of nodes must be removed to achieve disconnection, thus reflecting a considerably higher level of robustness in comparison to even n.
Finally, the rupture degree, which quantifies the impact of removing nodes on the graph's con-

nectivity, shows more negative values for odd n (i.e., $-2^{(n-1)} - 2$) and a less negative value for even n (i.e., -1), indicating that the graph is more vulnerable to disconnection when n is even.

Overall, these results convey that wrapped butterfly graphs are highly robust and stable structures, making them ideal for designing robust interconnected systems. Their high integrity, toughness, and tenacity, coupled with low scattering numbers and rupture degrees, ensure these networks can withstand significant disruptions while maintaining functionality. A similar methodology based on the topology of the network, can be utilized to identify vulnerabilities across different interconnection networks, including power grids, transportation systems, and communication networks, to gain insights into their vulnerabilities and performance in real-world scenarios.

6 Conclusion

The wrapped butterfly graphs high symmetry and unique structural properties make it an intriguing subject for studying network vulnerability. By identifying key vulnerability parameters, we have laid the groundwork for understanding its robustness. In the future, we would like to extend this analysis to other symmetric networks, to evaluate their vulnerability under various conditions. This exploration will not only enhance our theoretical understanding but also have practical implications for designing robust and reliable interconnection networks for diverse applications.

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